On Generalized $\overset{+}{H}{}^{v}$ − Non− Symmetric Recurrent Finsler Space F_n^*

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Abstract

In this present paper we introduced a Finsler space F_n^* for which the curvature tensor $\overset{+}{H}{}^i_{jkh}$ satisfied the generalized recurrence property with respect to non−symmetric connection parameter $\left(\Gamma^{*i}_{kh} \neq \Gamma^{*i}_{hk}\right)$ which given by the condition $\overset{+}{H}{}^i_{jkh}\big|_\ell = \lambda_\ell \overset{+}{H}{}^i_{jkh} + \mu_\ell\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right)$, $\overset{+}{H}{}^i_{jkh} \neq 0$, where $\big|_\ell$ is the $v-$ covariant differential operator, λ_ℓ and μ_ℓ are non−zero covariant vectors field and such space is called a generalized $\overset{+}{H}{}^{v}-$ recurrent space , denoted it briefly by $G\overset{+}{H}{}^{v} - RF_n^*$.

The purpose of this paper is to develop the above space by (i) obtaining the $v-$covariant derivative for the torsion tensor $\overset{+}{H}{}^i_{kh}$ and the deviation tensor $\overset{+}{H}{}^i_h$ in non−symmetric space, (ii) to prove that Ricci tensor $\overset{+}{H}_{jk}$, the curvature vector $\overset{+}{H}_k$ and the curvature scalar $\overset{+}{H}$ are non vanishing in our space, (iii) to prove that the tensor $(\overset{+}{H}_{hk} = \overset{+}{H}_{kh})$ behaves as recurrent in $G\overset{+}{H}{}^{v} - RF_n^*$ and (iv) to discuss the possibilities forms of decomposition for the curvature tensor $\overset{+}{H}{}^i_{jkh}$, we established the decomposition of the curvature tensor $\overset{+}{H}{}^i_{jkh}$ in a Finsler space F_n equipped with non−symmetric connection parameter.

***Keywords**: Generalized $\overset{+}{H}{}^{v}-$ recurrent space, decomposition of the curvature tensor$\overset{+}{H}{}^i_{jkh}$.*

1.Introduction

C. K. Mishra and G. Lodhi [4] discussed $C^h -$ *recurrent* and $C^v -$ *recurrent* Finsler space of second order and obtained different theorems regarding these spaces, also discussed the decomposability of the curvature tensor in recurrent conformal Finsler spaces. The decomposition of the curvature tensor $\overset{+}{R}{}^i_{jkh}$ in a Finsler space F_n equipped with non−connection parameter studied by P. Mishra, K. Srivistava and S. B. Mishra [5].

Let us consider an $n-$dimensional Finsler space F_n equipped with the metric function F satisfying the requesite condition [7].

Let consider the components of the corresponding metric tensor g^*_{ij} , Cartan's connection parameter Γ^{*i}_{jk}. These are symmetric in their lower indices and positively homogeneous of degree zero in the directional argument .

The two sets of quantities g_{ij} and its associate tensor g^{ij} are related by

(1.1) $$g_{ij}\, g^{jk} = \delta^k_i = \begin{cases} 1 & if \quad i = k \ , \\ 0 & if \quad i \neq k \ . \end{cases}$$

* The indices $i, j, k, \ldots$ assume positive integer values from 1 to n.

The vectors y_i and y^i are related by the relation

(1.2) a) $y_j = g_{ij} y^i$ and b) $\dot{\partial}_i y_j = g_{ij}$.

The tensor C^*_{ijk} defined by

(1.3) $$C_{ijk} := \frac{1}{2} \dot{\partial}_i \, g_{jk} = \frac{1}{4} \dot{\partial}_i \, \dot{\partial}_j \, \dot{\partial}_k \, F^2,$$

is known as $(h)hv-$*torsion tensor* [3]. It is positively homogenous of degree -1 in the directional argument and symmetric in all its indices.

The $(v)hv-$torsion tensor C^h_{ik} and its associate $(h)hv-$torsion tensor C_{ijk} are related by

(1.4) a) $C^i_{jk} y^i = 0 = C^i_{kj} y^i$, b) $y_i C^i_{jk} = 0$ and c) $C_{ijk} = g_{hj} C^h_{ik}$.

The $(v)hv-$torsion tensor C^h_{ik} is also positively homogenous of degree -1 in the directional argument and symmetric in its lower indices.

$E'-$Cartan deduced the $v-$covariant derivative for an arbitrary vector field X^i with respect to y^k [2]

(1.5) $$X^i \big|_k := \dot{\partial}_k X^i + X^r C^{\,i}_{rk} \, .$$

In view of (1.4b) and (1.5), we have

(1.6) a) $y^i \big|_k = \delta^i_k$, b) $y_i \big|_k = g_{ki}$ and c) $g_{ij} \big|_k = 0$.

2. On Study of Generalized $\overset{+}{H}{}^v$ – Non– Symmetric Recurrent Space

G. H. Vranceam [7] has defined a non$-$symmetric connection $\left(\Gamma^{*i}_{jk} \neq \Gamma^{*i}_{kj}\right)$ in $n-$ dimensional Finsler space F_n.

Let consider an $n-$ dimensional Finsler space F_n with non$-$ symmetric connection $\left(\Gamma^{*i}_{jk} \neq \Gamma^{*i}_{kj}\right)$ which is based on a non $-$ symmetric fundamental tensor $g_{ij} \neq g_{ji}$.

Let write

(2.1) $$\Gamma^{*i}_{jk} = M^{*i}_{jk} + \tfrac{1}{2} N^{*i}_{jk} \, ,$$

where M^{*i}_{jk} and $\frac{1}{2} N^{*i}_{jk}$ are respectively the symmetric and skew$-$symmetric parts of Γ^{*i}_{jk}.

We introduce another connection parameter Γ^{*i}_{kj} defined as order

(2.2) $$\bar{\Gamma}^{*i}_{kj} = M^{*i}_{kj} - \tfrac{1}{2} N^{*i}_{kj} \, .$$

With the help of (2.1) and (2.2), we get

$$\Gamma^{*i}_{jk} = \bar{\Gamma}^{*i}_{jk} \, .$$

Following $E'-$ Cartan [2], let a vertical stroke $\big|_j$, follow by an index denote covariant derivative with respect to y^j , the covariant derivative of any contavariant vector field X^i with respect to y^j is defined as follows:

(2.3) $$X^i \big|_k := \dot{\partial}_j X^i + X^r C^{\,i}_{rj} \, ,$$

where a positive sign below an index and following by a vertical stroke indicates that the covariant derivative has been formed with respect to the connection Γ^{*i}_{kj} as for as that index is concerned

* Unless stated otherwise . Hence forth all geometric objects are to be function of line-elements.

The covariant derivative defined in (2.3) is called $\oplus$ *−covariant differentiation of* X^i *with respect to* y^j, also is called $v -$ *covariant differentiation (Cartan's covariant differentiation of the first kind*).

The entity $\overset{+}{H}{}^i_{jkh}$ is called *the curvature tensor* (*with respect the* $\oplus$ *−covariant derivative*) of Finsler space with respect to the non−symmetric connection Γ^{*i}_{jk}, such that

$$\overset{+}{H}{}^i_{jkh} := \dot\partial_h G^i_{jk} + G^r_{jk}\, G^i_{rh} + G^i_{rjh}\, G^r_k - \dot\partial_k G^i_{jh} - G^r_{jh} G^i_{rk} - G^i_{rjk}\, G^r_h .$$

We shall use the following identities, notations and contractions for H^{+i}_{jkh}

(2.4) a) $\overset{+}{H}{}^i_{jkh}\, y^j = \overset{+}{H}{}^i_{kh}$, b) $\overset{+}{H}{}^i_{kh}\, y^k = \overset{+}{H}{}^i_h$, c) $\overset{+}{H}{}^i_{jki} = \overset{+}{H}_{jk}$, d) $\overset{+}{H}{}^i_i = (n-1)\overset{+}{H}$,

e) $\overset{+}{H}{}^i_{ki} = \overset{+}{H}_k$, f) $\overset{+}{H}{}^i_{iki} = \overset{+}{H}_{hk} - \overset{+}{H}_{kh}$, g) $\overset{+}{H}_{jk}\, y^k = (n-1)\dot\partial_j \overset{+}{H} - \overset{+}{H}_j$

and h) $\overset{+}{H}_k\, y^k = (n-1)\overset{+}{H}$.

Hence forth a Finsler space equipped with non−symmetric connection will be denoted by F_n^+ .

A Finsler space F_n^* is said to be *a generalized* $\overset{+}{H}{}^v$ *− non− symmetric recurrent space* for which the curvature tensor H^{+i}_{jkh} satisfies the following condition

(2.5) $$\overset{+}{H}{}^i_{jkh}\,|_\ell = \lambda_\ell\, \overset{+}{H}{}^i_{jkh} + \mu_\ell(\delta^i_h g_{jk} - \delta^i_k g_{jh}), \qquad \overset{+}{H}{}^i_{jkh} = 0.$$

We shall denote it briefly by $GH^{+v} - RF_n^*$ and the tensor which satisfies the condition (2.5) will be called *a generalized recurrent*, where λ_ℓ and μ_ℓ are non−zero covariant vectors field.

Let us consider an $G\overset{+}{H}{}^v - RF_n^*$ which is characterized by the condition (2.5). Transvecting the condition (2.5) by y^j, using (2.4a), (1.6a) and (1.2a), we get

(2.6) $$\overset{+}{H}{}^i_{kh}\,|_\ell = \lambda_\ell\, \overset{+}{H}{}^i_{kh} + \overset{+}{H}{}^i_{\ell kh} + \mu_\ell(\delta^i_h y_k - \delta^i_k y_h).$$

Thus, we conclude

Theorem 2.1. In $G\overset{+}{H}{}^v - RF_n^*$, the v −covariant derivative of first order for the torsion tensor H^{+i}_{kh} is given by (2.6).

The equation (2.6) can be written as

(2.7) $$\overset{+}{H}{}^i_{\ell kh} = \overset{+}{H}{}^i_{kh}\,|_\ell - \lambda_\ell\, \overset{+}{H}{}^i_{kh} - \mu_\ell(\delta^i_h y_k - \delta^i_k y_h).$$

Thus, we conclude

Theorem 2.2. In $G\overset{+}{H}{}^v - RF_n^*$, the curvature tensor $\overset{+}{H}{}^i_{\ell kh}$ is defined by (2.7).

Contracting the indices i and h in (2.5), using (2.1d) and in view of (1.1), we get

(2.8) $$\overset{+}{H}_{jk}\,|_\ell = \lambda_\ell\, \overset{+}{H}_{jk} + (n-1)\mu_\ell g_{jk} .$$

The equation (2.8) shows that Ricci tensor $\overset{+}{H}_{jk}$ can't vanish, since the vanishing of it would implies the vanishing of the covariant vector field, $.e.$ $\mu_\ell = 0$, a contraction.

Thus, we conclude

Theorem 2.3. In $G\overset{+}{H}{}^{v} - RF_n^*$, Ricci tensor $\overset{+}{H}_{jk}$ is non−vanishing

3. Decomposition of the curvature tensor $\overset{+}{H}{}^{i}_{jkh}$ in a Finsler Space Equipped with Non−Symmetric Connection

We shall discuss some of the decompositions of the curvature tensor $\overset{+}{H}{}^{i}_{jkh}$ in a Finsler space equipped with non−symmetric connection for Berwald curvature tensor.

Now, let us consider the decomposition of the curvature tensor $\overset{+}{H}{}^{i}_{jkh}$ in a Finsler space F_n^***,** since the curvature tensor under consideration is a mixed tensor of rank 4, hence it may be written either as a tensor product of a vector and a tensor of rank 3 or as a tensor product of two tensors each of rank 2.

In the first case, the possibilities form of decomposition for the curvature tensor $\overset{+}{H}{}^{i}_{jkh}$ are as follows:

(3.1) a) $\overset{+}{H}{}^{i}_{jkh} = \overset{+}{X}{}^{i}\, \overset{+}{\Psi}_{jkh}$, b) $\overset{+}{H}{}^{i}_{jkh} = \overset{+}{X}_{j} \overset{+}{\Psi}{}^{i}_{kh}$,
c) $\overset{+}{H}{}^{i}_{jkh} = \overset{+}{X}_{k} \overset{+}{\Psi}{}^{i}_{jh}$ and d) $\overset{+}{H}{}^{i}_{jkh} = \overset{+}{X}_{h} \overset{+}{\Psi}{}^{i}_{jk}$.

In the second case, the possibilities as follows:

(3.2) a) $\overset{+}{H}{}^{i}_{jkh} = \overset{+}{Y}{}^{i}_{j} \overset{+}{\Phi}_{kh}$, b) $\overset{+}{H}{}^{i}_{jkh} = \overset{+}{Y}{}^{i}_{k} \overset{+}{\Phi}_{jh}$ and c) $\overset{+}{H}{}^{i}_{jkh} = \overset{+}{Y}{}^{i}_{h} \overset{+}{\Phi}_{jk}$
.

Out of several possibilities given by (3.1a), (3.1b), (3.1c), (3.1d), (3.2a), (3.2b) and (3.2c), our goal is to study the possibilities given by (3.1a), (3.1b) and (3.2a).

Let us consider a Finsler space F_n^* whose curvature tensor $\overset{+}{H}{}^{i}_{jkh}$ is decomposable in the form (3.1a).

Taking the v −covariant derivative for the form (3.1a) with respect to y^{ℓ}, we get

(3.3) $\overset{+}{H}{}^{i}_{jkh}\,|_{\ell} = \overset{+}{X}{}^{i}\,|_{\ell}\, \overset{+}{\Psi}_{jkh} + \overset{+}{X}{}^{i}\, \overset{+}{\Psi}_{jkh}\,|_{\ell}$.

In view of the condition (2.5) and (3.3), we get

(3.4) $\lambda_\ell \overset{+}{H}{}^{i}_{jkh} + \mu_\ell\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) = \overset{+}{X}{}^{i}\,|_{\ell}\, \overset{+}{\Psi}_{jkh} + \overset{+}{X}{}^{i}\, \overset{+}{\Psi}_{jkh}\,|_{\ell}$.

By using (3.1a) and if the decomposable vector field $\overset{+}{X}{}^{i}$ is covariant constant, (3.4) can be written as

$$\lambda_\ell \overset{+}{X}{}^{i}\, \overset{+}{\Psi}_{jkh} + \mu_\ell\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) = \overset{+}{X}{}^{i}\, \overset{+}{\Psi}_{jkh|\ell}$$

or

(3.5) $\overset{+}{\Psi}_{jkh}|_{\ell} = \lambda_\ell \overset{+}{\Psi}_{jkh} + \eta_{\ell i}\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right)$,

where $\eta_{\ell i} = \frac{\mu_\ell}{\overset{+}{X}{}^{i}}$.

Thus, we conclude

Theorem 3.1. *In* $G\overset{+}{H}{}^{v} - RF_n^*$, *if the curvature tensor* $\overset{+}{H}{}^{i}_{jkh}$ *is decomposable in the form (3.1a), then the decomposable tensor field* $\overset{+}{\Psi}_{jkh}$ *is generalized recurrent, provided that the decomposable vector field* $\overset{+}{X}{}^{i}$ *is covariant constant.*

Transvecting (3.5) by y^j , using (1.6a), (1.2a) and in view of (1.1), we get

(3.6) $\overset{+}{\Psi}_{kh}|_{\ell} = \lambda_\ell \overset{+}{\Psi}_{kh} + \overset{+}{\Psi}_{\ell kh} + \eta_{\ell i}\left(\delta^i_h y_k - \delta^i_k y_h\right)$,

where $\overset{+}{\Psi}_{kh} = \overset{+}{\Psi}_{jkh} y^j$.
Thus, we conclude

Theorem 3.2. *In $G\overset{+}{H}{}^v - RF_n^*$, if the curvature tensor $\overset{+}{H}{}^i_{jkh}$ is decomposable in the form (3.1a), then the v −covariant derivative of first order for the tensor field $\overset{+}{\Psi}_{kh}$ is given by (3.6), provided that the decomposable vector field $\overset{+}{X}{}^i$ is covariant constant.*

The equation (3.6) can be written as

(3.7) $\overset{+}{\Psi}_{\ell kh} = \overset{+}{\Psi}_{kh}|_\ell - \lambda_\ell \overset{+}{\Psi}_{kh} - \eta_{\ell i}\left(\delta^i_h y_k - \delta^i_k y_h\right).$

Thus, we conclude

Theorem 3.3. *In $G\overset{+}{H}{}^v - RF_n^*$, if the curvature tensor $\overset{+}{H}{}^i_{jkh}$ is decomposable in the form (3.1a), then the decomposable tensor field $\overset{+}{\Psi}_{\ell kh}$ is defined by (3.7), provided that the decomposable vector field $\overset{+}{X}{}^i$ is covariant constant.*

If the tensor $\overset{+}{\Psi}_{kh}$ is recurrent, then (3.7) can be written as

(3.8) $\overset{+}{\Psi}_{\ell kh} = \omega_{\ell kh} - \omega_{\ell hk}$,

where $\omega_{\ell kh} = \eta_{\ell i}\, \delta^i_k y_h$ and $\omega_{\ell hk} = \eta_{\ell i}\, \delta^i_h y_k$.
Thus, we conclude

Theorem 3.4. *In $G\overset{+}{H}{}^v - RF_n^*$, if the curvature tensor $\overset{+}{H}{}^i_{jkh}$ is decomposable in the form (3.1a) and the tensor field $\overset{+}{\Psi}_{kh}$ is recurrent, then the tensor field $\overset{+}{\Psi}_{\ell kh}$ is defined by (3.8), provided that the decomposable vector field $\overset{+}{X}{}^i$ is covariant constant.*

If the tensor $\omega_{\ell kh}$ is skew−symmetri in the last two indices, then (3.8) becomes

(3.9) $\overset{+}{\Psi}_{\ell kh} = 2\omega_{\ell kh}.$

Thus, we conclude

Theorem 3.5. *In $G\overset{+}{H}{}^v - RF_n^*$, if the curvature tensor $\overset{+}{H}{}^i_{jkh}$ is decomposable in the form (3.1a) and the tensor field $\omega_{\ell kh}$ is skew−symmetri in the last two indices, then the tensor field $\overset{+}{\Psi}_{\ell kh}$ is defined by (3.9), provided that the decomposable vector field $\overset{+}{X}{}^i$ is covariant constant.*

Let us consider a Finsler space F_n^* whose curvature tensor $\overset{+}{H}{}^i_{jkh}$ is decomposable in the form (3.1b).

Taking the v −covariant derivative for the form (3.1b) with respect to y^ℓ, we get

(3.10) $\overset{+}{H}{}^i_{jkh}|_\ell = \overset{+}{X}_j|_\ell \overset{+}{\Psi}{}^i_{kh} + \overset{+}{X}_j \overset{+}{\Psi}{}^i_{kh}\,|_\ell.$

In view of the condition (2.5) and (3.10), we get

(3.11) $\lambda_\ell \overset{+}{H}{}^i_{jkh} + \mu_\ell\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) = \overset{+}{X}_j|_\ell \overset{+}{\Psi}{}^i_{kh} + \overset{+}{X}_j \overset{+}{\Psi}{}^i_{kh}\,|_\ell$.

By using (3.1b) and if the decomposable vector field $\overset{+}{X}_j$ is covariant constant, then (3.11) can be written as

$$\lambda_\ell \overset{+}{X}_j \overset{+}{\Psi}{}^i_{kh} + \mu_\ell\left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right) = \overset{+}{X}_j \overset{+}{\Psi}{}^i_{kh|\ell}$$

or

(3.12) $\overset{+}{\Psi}{}^i_{kh}|_\ell = \lambda_\ell \overset{+}{\Psi}{}^i_{kh} + \eta^i_\ell \left(\delta^i_h g_{jk} - \delta^i_k g_{jh}\right)$,

where $\eta_{\ell i} = \frac{\mu_\ell}{\overset{+}{X}_j}$.

Thus, we conclude

Theorem 3.6. *In $\overset{+}{GH}{}^v - RF^*_n$, if the curvature tensor $\overset{+}{H}{}^i_{jkh}$ is decomposable in the form (3.1b), then the decomposable tensor field $\overset{+}{\Psi}{}^i_{kh}$ is generalized recurrent, provided that the decomposable vector field $\overset{+}{X}_j$ is covariant constant.*

The equation (3.12) can be written

(3.13) $\overset{+}{\Psi}{}^i_{kh}|_\ell = \lambda_\ell \overset{+}{\Psi}{}^i_{kh} + \left(\theta^i_{\ell hk} - \theta^i_{\ell kh}\right)$,

where $\theta^i_{\ell hk} = \eta^i_\ell \delta^i_h g_{jk}$ and $\theta^i_{\ell kh} = \eta^i_\ell \delta^i_k g_{jh}$.

Now, if the tensor field $\theta^i_{\ell hk}$ is symmetric in the last two indices, then (3.13) can be written as

$$\overset{+}{\Psi}{}^i_{kh}|_\ell = \lambda_\ell \overset{+}{\Psi}{}^i_{kh} .$$

Thus, we conclude

Theorem 3.7. *In $\overset{+}{GH}{}^v - RF^*_n$, if the curvature tensor $\overset{+}{H}{}^i_{jkh}$ is decomposable in the form (3.1b) and the tensor field $\theta^i_{\ell hk}$ is symmetric in the last two indices, then the decomposable tensor field $\overset{+}{\Psi}{}^i_{kh}$ is recurrent, provided that the decomposable vector field $\overset{+}{X}_j$ is covariant constant.*

If the tensor field $\theta^i_{\ell hk}$ is skew−symmetric in the last two indices, then (3.13) can be written

(3.14) $\overset{+}{\Psi}{}^i_{kh}|_\ell = \lambda_\ell \overset{+}{\Psi}{}^i_{kh} + 2\theta^i_{\ell hk}$.

Thus, we conclude

Theorem 3.8. *In $\overset{+}{GH}{}^v - RF^*_n$, if the curvature tensor $\overset{+}{H}{}^i_{jkh}$ is decomposable in the form (3.1b) and the tensor field $\theta^i_{\ell hk}$ is skew−symmetri in the last two indices, then the v −covariant derivative of first order for the decomposable tensor field $\overset{+}{\Psi}{}^i_{kh}$ is given by (3.14), provided that the decomposable vector field $\overset{+}{X}_j$ is covariant constant.*

Transvecting (3.12) by y_i , using (1.6b) and in view of (1.1), we get

(3.15) $\overset{+}{\Psi}_{kh}|_\ell = \lambda_\ell \overset{+}{\Psi}_{kh} + \overset{+}{\Psi}_{kh\ell} + \left(\pi_{\ell hk} - \pi_{\ell kh}\right)$,

where $\overset{+}{\Psi}_{kh} = \overset{+}{\Psi}{}^i_{kh} y_i$, $\overset{+}{\Psi}_{\ell kh} = \overset{+}{\Psi}{}^i_{kh} g_{\ell i}$, $\pi_{\ell hk} = \eta^i_\ell \delta^i_h g_{jk} y_i$ and $\pi_{\ell kh} = \eta^i_\ell \delta^i_k g_{jh} y_i$.

Thus, we conclude

Theorem 3.9. *In $\overset{+}{GH}{}^v - RF^*_n$, if the curvature tensor $\overset{+}{H}{}^i_{jkh}$ is decomposable in the form (3.1b), then the v −covariant derivative of first order for the tensor field $\overset{+}{\Psi}_{kh}$ is given by (3.15), provided that the decomposable vector field $\overset{+}{X}_j$ is covariant constant.*

The equation (3.15) can be written as

(3.16) $\overset{+}{\Psi}_{kh\ell} = \overset{+}{\Psi}_{kh}|_{\ell} - \lambda_{\ell}\overset{+}{\Psi}_{kh} + (\pi_{\ell hk} - \pi_{\ell kh})$.

Thus, we conclude

Theorem 3.10. *In $G\overset{+}{H}{}^{v} - RF_n^*$, if the curvature tensor $\overset{+}{H}{}^{i}_{jkh}$ is decomposable in the form (3.1b), then the decomposable tensor field $\overset{+}{\Psi}_{kh\ell}$ is defined by (3.16), provided that the decomposable vector field $\overset{+}{X}_j$ is covariant constant.*

Now, if the tensor field $\pi_{\ell hk}$ is symmetric in the last two lower indices h and k , then the equation (3.16), shows that the tensor $\overset{+}{\Psi}_{kh}$ can't be a recurrent, otherwise the tensor $\overset{+}{\Psi}_{kh\ell}$ will be vanish.

Thus, we conclude

Theorem 3.11. *In $G\overset{+}{H}{}^{v} - RF_n^*$, if the curvature tensor $\overset{+}{H}{}^{i}_{jkh}$ is decomposable in the form (3.1b) and the tensor field $\pi_{\ell hk}$* is symmetric in the last two lower indices , *then the tensor $\overset{+}{\Psi}_{kh}$ can't be a recurrent, provided that the decomposable vector field $\overset{+}{X}_j$ is covariant constant.*

If the tensor $\overset{+}{\Psi}_{kh}$ is recurrent, then (3.16) can be written as

(3.17) $\overset{+}{\Psi}_{kh\ell} = \pi_{\ell hk} - \pi_{\ell kh}$.

Thus, we conclude

Theorem 3.12. *In $G\overset{+}{H}{}^{v} - RF_n^*$, if the curvature tensor $\overset{+}{H}{}^{i}_{jkh}$ is decomposable in the form (3.1b) and the tensor field $\overset{+}{\Psi}_{kh}$ is recurrent, then the tensor field $\overset{+}{\Psi}_{kh\ell}$ is defined by (3.17), provided that the decomposable vector field $\overset{+}{X}_j$ is covariant constant.*

If the tensor $\pi_{\ell hk}$ is skew−symmetri in the last two indices, then (3.17) becomes

(3.18) $\overset{+}{\Psi}_{kh\ell} = 2\pi_{\ell hk}$.

Thus, we conclude

Theorem 3.13. *In $G\overset{+}{H}{}^{v} - RF_n^*$, if the curvature tensor $\overset{+}{H}{}^{i}_{jkh}$ is decomposable in the form (3.1b), the tensor field $\overset{+}{\Psi}_{kh}$ is recurrent and the tensor field $\pi_{\ell hk}$* is skew − symmetric in the last two indices, then the tensor field *$\overset{+}{\Psi}_{kh\ell}$ is defined by (3.18), provided that the tensor field $\overset{+}{\Psi}_{kh}$ is recurrent and the decomposable vector field $\overset{+}{X}_j$ is covariant constant.*

Let us consider a Finsler space F_n^* whose curvature tensor $\overset{+}{H}{}^{i}_{jkh}$ is decomposable in the form (3.2a).

Taking the v −covariant derivative for the form (3.2a) with respect to y^{ℓ}, we get

(3.19) $\overset{+}{H}{}^{i}_{jkh}\,|_{\ell} = \overset{+}{\Upsilon}{}^{i}_{j}\,|_{\ell}\overset{+}{\Phi}_{kh} + \overset{+}{\Upsilon}{}^{i}_{j}\,\overset{+}{\Phi}_{kh}\,|_{\ell}$.

In view of the condition (2.5) and (3.19), we get

$$\lambda_{\ell}\,\overset{+}{H}{}^{i}_{jkh} + \mu_{\ell}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) = \overset{+}{\Upsilon}{}^{i}_{j}\,|_{\ell}\overset{+}{\Phi}_{kh} + \overset{+}{\Upsilon}{}^{i}_{j}\,\overset{+}{\Phi}_{kh}\,|_{\ell}\,.$$

By using (3.2a) and if the decomposable tensor field $\overset{+}{\Upsilon}{}_j^i$ is covariant constant, then the above equation can be written as

$$\lambda_\ell \overset{+}{\Upsilon}{}_j^i \overset{+}{\Phi}_{kh} + \mu_\ell (\delta_h^i g_{jk} - \delta_k^i g_{jh}) = \overset{+}{\Upsilon}{}_j^i \overset{+}{\Phi}_{kh|\ell}$$

which implies

(3.20) $\qquad \overset{+}{\Phi}_{kh}|_\ell = \lambda_\ell \overset{+}{\Phi}_{kh} + \eta_{\ell i}^j (\delta_h^i g_{jk} - \delta_k^i g_{jh})$,

where $\eta_{\ell i}^j = \frac{\mu_\ell}{\overset{+}{\Upsilon}{}_j^i}$.

Thus, we conclude

Theorem 3.14. *In $G\overset{+}{H}{}^v - RF_n^*$, if the curvature tensor $\overset{+}{H}{}_{jkh}^i$ is decomposable in the form (3.2a), then the decomposable tensor field $\overset{+}{\Phi}_{kh}$ is generalized recurrent, provided that the decomposable tensor field $\overset{+}{\Upsilon}{}_j^i$ is covariant constant.*

The equation (3.20) can be written

(3.21) $\qquad \overset{+}{\Phi}_{kh}|_\ell = \lambda_\ell \overset{+}{\Phi}_{kh} + (\upsilon_{\ell hk} - \upsilon_{\ell kh})$,

where $\upsilon_{\ell hk} = \eta_{\ell i}^j \delta_h^i g_{jk}$ and $\upsilon_{\ell kh} = \eta_{\ell i}^j \delta_k^i g_{jh}$.

Thus, we conclude

Theorem 3.15. *In $G\overset{+}{H}{}^v - RF_n^*$, if the curvature tensor $\overset{+}{H}{}_{jkh}^i$ is decomposable in the form (3.2a), then the $v-$covariant derivative of first order for the decomposable tensor field $\overset{+}{\Phi}_{kh}$ is given by (3.21), provided the decomposable tensor field $\overset{+}{\Upsilon}{}_j^i$ is covariant constant.*

Now, if the tensor field $\upsilon_{\ell hk}$ is skew−symmetri in the last two indices h and , we get

(3.22) $\qquad \overset{+}{\Phi}_{kh}|_\ell = \lambda_\ell \overset{+}{\Phi}_{kh} + 2\upsilon_{\ell hk}$.

Thus, we conclude

Theorem 3.16. *In $G\overset{+}{H}{}^v - RF_n^*$, if the curvature tensor $\overset{+}{H}{}_{jkh}^i$ is decomposable in the form (3.2a) and the tensor field $\upsilon_{\ell hk}$ is skew−symmetri in the last two indices, then the $v-$covariant derivative of first order for the decomposable tensor field $\overset{+}{\Phi}_{kh}$ is given by (3.22), provided the decomposable tensor field $\overset{+}{\Upsilon}{}_j^i$ is covariant constant.*

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